Kähler manifolds

particular kinds of complex manifolds with additional structure.

$\boxed{\text{Example}}$ $\mathbb{R}^{2n} = \{(x_1, y_1, x_2, y_2, \dots, x_n, y_n) : \begin{smallmatrix} x_j \in \mathbb{R} \\ y_j \in \mathbb{R} \end{smallmatrix}\ \forall j\}$

Complex Structure At each point p of $\mathbb{R}^{2n}$,

The tangent space $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ ← I'll use the same coordinates for the vectors.

$(\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_n}, \partial_{y_n})$

A linear map on $T_p\mathbb{R}^{2n}$

An almost complex structure

$J : T_p\mathbb{R}^{2n} \to T_p\mathbb{R}^{2n}$ a linear transformation

with matrix

$$\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$J(\partial_{x_1}) = \partial_{y_1}$$
$$J(\partial_{y_1}) = -\partial_{x_1}$$

$$J^2 = -I$$

complexified tangent space. $T_p\mathbb{R}^{2n} \otimes \mathbb{C}$:

eigenvalues of J are $\pm i$

The $+i$ eigenspace is spanned by $\{\partial_{z_1}, \partial_{z_2}, \dots, \partial_{z_n}\}$

where $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$

(in complex analysis $\frac{\partial}{\partial z} = \partial_z = \frac{1}{2}\left(\underset{\uparrow \partial_x}{\frac{\partial}{\partial x}} - i\underset{\uparrow \partial_y}{\frac{\partial}{\partial y}}\right)$)

The $-i$ eigenspace is spanned

$\{\overline{\partial_{z_j}} : 1 \le j \le n\}$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} +i \\ 1 \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix}. \checkmark$$

$$T_p^{1,0}\mathbb{R}^{2n} = \operatorname{span}_{\mathbb{C}}\{\partial_{z_j} : 1 \le j \le n\}$$ holomorphic tangent space

$$T_p^{0,1}\mathbb{R}^{2n} = \operatorname{span}_{\mathbb{C}}\{\overline{\partial}_{z_j} : 1 \le j \le n\}$$ anti hol. tangent space

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Differential forms on $\mathbb{R}^{2n}$ can be written in terms of dx_j's & dy_j's or

$$\left(dz_j(\partial_{z_k}) = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{otherwise} \end{cases} \right.$$

$$dz_j = dx_j + i\,dy_j$$

$$\frac{1}{2}(dz_j + d\overline{z}_j) = dx_j$$

$$\frac{1}{2i}(dz_j - d\overline{z}_j) = dy_j$$

So we can write all differential forms (with complex coeffs) in terms of dz_j's & $d\overline{z}_j$'s.

Exterior derivative

$$d : \underbrace{\Omega^*(\mathbb{R}^{2n})}_{\substack{\text{(complex) differential} \\ \text{forms on } \mathbb{R}^{2n}}} \longrightarrow \Omega^{*+1}(\mathbb{R}^{2n})$$

$$d = \sum dx_j \wedge \frac{\partial}{\partial x_j} + \sum dy_j \wedge \frac{\partial}{\partial y_j}$$

$$= \underbrace{\sum dz_j \wedge \frac{\partial}{\partial z_j}}_{\partial} + \underbrace{\sum d\overline{z}_j \wedge \frac{\partial}{\partial \overline{z}_j}}_{\overline{\partial}}$$

$d = \partial + \bar\partial \qquad V, W \in T_p\mathbb{R}^{2n}$

metric $g(V,W) = \langle V, W \rangle \underset{\mathbb{R}^{2n}}{=} V \cdot W = W \cdot$

symmetric

fundamental form (2-form) $\omega(V,W) = g(JV, W)$

how does a 2-form eat two vector fields & get a fcn?

$(dx \wedge dw)(a\partial_x + b\partial_y + c\partial_w, d\partial_x + e\partial_y + f\partial_w)$

$= dx(a\partial_x + b\partial_y + c\partial_w) dw(d\partial_x + e\partial_y + f\partial_w)$

$- dw(a\partial_x + b\partial_y + c\partial_w) dx(d\partial_x + e\partial_y + f\partial_w)$

$= a \cdot f - c \cdot d.$

Notice $g(JW, V) = g(V, JW) = g(IV, JW)$

$= -g(J^2 V, JW)$

J is compatible with the metric g

$\Leftrightarrow g(JV, JW) = g(V, W)$

$\Leftrightarrow \mathbb{R}^{2n}$: J is an orthogonal matrix.

$\to g(JW, V) = -g(JV, W).$

$\omega(\cdot, \cdot)$ is antisymmetric $\Rightarrow$ a 2-form.

recall $J(\partial_{x_j}) = \partial_{y_j}, \; J(\partial_{y_j}) = -\partial_{x_j}$

$$\omega(\partial_{x_j}, \partial_{y_j}) = g(J(\partial_{x_j}), \partial_{y_j}) = g(\partial_{y_j}, \partial_{y_j}) = 1$$

$$\omega(\partial_{x_j}, \underset{\text{else}}{\text{anything}}) = 0$$

$$\omega(\partial_{y_j}, \partial_{x_j}) = -1$$

$$\Rightarrow \boxed{\omega = \sum_{j=1}^{n} dx_j \wedge dy_j}$$ Fundamental form.

$$\boxed{\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j}$$

in general, if $g(\cdot,\cdot)$ is a metric on $2n$ coordinates, you get

$$\omega = \frac{i}{2} \sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j$$

where h_{ij} is a Hermitian symmetric matrix at each point.

$$(h^* = \overline{(h^T)} = h)$$

Notice $\omega^n = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ copies}} = n!\, \underbrace{dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n}_{\text{volume form on } \mathbb{R}^{2n}.}$

$$T_p^{1,0} = \operatorname{span}_{\mathbb{C}}\{\partial_{z_j}\} \quad T_p^{0,1} = \operatorname{span}_{\mathbb{C}}\{\partial_{\bar{z}_j}\}.$$

J is called an <u>integrable almost complex structure</u> = <u>complex structure</u>

if $T^{1,0}$ & $T^{1,0}$ are <u>involutive</u> subbundles of $TM \otimes \mathbb{C}$.

↳ If X, Y are two vector fields in $\Gamma(T^{1,0})$, then $[X,Y]$ is also in $\Gamma(T^{1,0})$.

Our J is integrable/involutive $\Rightarrow$ we have a complex structure.

FYI: This condition is satisfied iff

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0$$

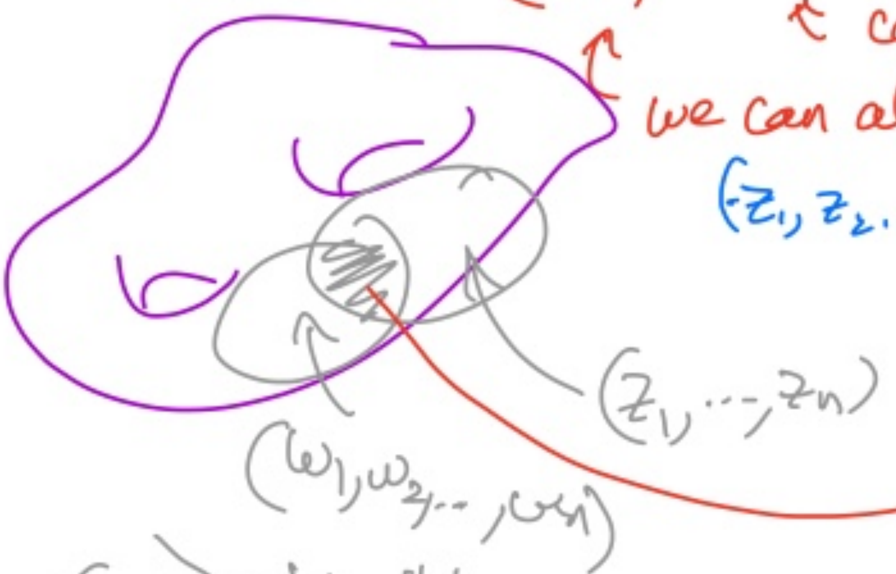
$\forall$ v. fields X, Y.

A complex manifold is

$$(M^{2n}, J)$$

J: complex structure

we can always find coordinates $(z_1, z_2, \dots, z_n)$ locally.



in overlap.

$\phi : \{(z_1, \dots, z_n)\} \to \{(w_1, \dots, w_n)\}$

is a biholomorphic map.

(M^{2n}, g, J) Hermitian manifold.

Kähler manifold:

$\underbrace{\underbrace{(M, J)}_{\text{complex}} + g \text{ compatible}}_{\text{Hermitian}} \Longrightarrow \omega$ fundamental

$+ \boxed{d\omega = 0}$ Kähler condition

ω: Kähler form.